Exercice 28

$$A = \mathcal{M}_B(f) = \frac{1}{6}\begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

① La matrice $A$ est symétrique à coefficients réels donc $\boxed{A \text{ est diagonalisable}}$

$$rg(A) = rg(6A) = rg\begin{pmatrix} \boxed{1} & 2 & 5 \\ -2 & 2 & 2 \\ 5 & -2 & 1 \end{pmatrix} = rg\begin{pmatrix} 1 & 2 & 5 \\ 0 & 6 & 12 \\ 0 & -12 & -24 \end{pmatrix}$$

$L_2 \leftarrow L_2 + 2L_1$
$L_3 \leftarrow L_3 - 5L_1$

$$= rg\begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}$$

$L_2 \leftarrow L_2/6$
$L_3 \leftarrow L_3/12$

$$= rg\begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 2 \Rightarrow 0 \in Sp(A) \text{ et } \underline{\dim E_0 = 3 - rg(A) = 1}$$

$$rg(A - I) = rg\,\frac{1}{6}\begin{pmatrix} -1 & -2 & 1 \\ -2 & -4 & 2 \\ 1 & 2 & -1 \end{pmatrix} = rg\begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1$$

Donc $1 \in Sp(A)$ et $\underline{\dim(E_1)} = 3 - rg(A - I) = \underline{3 - 1 = 2}.$

$$\dim(E_0) + \dim(E_1) = 3 = \dim(\mathbb{R}^3)$$

<u>Conclusion</u> $\boxed{Sp(A) = \{0, 1\}}$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_0 \Leftrightarrow \begin{cases} x + 2y + 5z = 0 \\ y + 2z = 0 \end{cases} \Leftrightarrow \begin{cases} y = -2z \\ x = -2y - 5z = -z \end{cases} \forall z \in \mathbb{R}$$

<u>Conclusion</u> $\boxed{E_0 = \left\{\begin{pmatrix} -z \\ -2z \\ z \end{pmatrix}, z \in \mathbb{R}\right\} = Vect\left\{\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right\}}$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_1 \Leftrightarrow (A - I)X = 0 \Leftrightarrow -x - 2y + z = 0$$
$$\Leftrightarrow z = x + 2y, \forall x, y \in \mathbb{R}$$

<u>Conclusion</u> $\boxed{E_1 = \left\{\begin{pmatrix} x \\ y \\ x + 2y \end{pmatrix}, x, y \in \mathbb{R}\right\} = Vect\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\}}$

(2) $0 \in Sp(A) \iff \operatorname{Ker} A = E_0 \neq \{0\} \iff$ $\boxed{A \text{ non inversible}}$

(3)

```python
def proj(M):
    N = np.dot(M,M)
    Cpt = 0
    for i in range(3):
        for j in range(3):
            if np.abs(N[i,j]-M[i,j])<1e-5:
                Cpt += 1
    return Cpt == 9
```

ou alors, si on connaît la fonction 'np.allclose()' :

```python
def proj(M):
    return np.allclose(np.dot(M,M), M)
```

(4) Méthode 1 : la plus rapide !

- $E_1 \subset \operatorname{Im} f$ $[x \in E_1 \iff f(x) = x \implies x \in \operatorname{Im} f]$
- $\dim(E_1) = 2$ [d'après (1)] et $\operatorname{rg}(f) = 3 - \dim \operatorname{Ker} f$
$= 3 - \dim E_0 = 2$

Donc $\dim(E_1) = \dim(\operatorname{Im} f)$ ; concl : $\boxed{E_1 = \operatorname{Im} f}$

Méthode 2 : $\operatorname{Im} f = \operatorname{Vect}\{f(e_1), f(e_2), f(e_3)\}$

$$= \operatorname{Vect}\left\{\begin{pmatrix}5\\-2\\1\end{pmatrix}, \begin{pmatrix}-2\\2\\2\end{pmatrix}, \begin{pmatrix}1\\2\\5\end{pmatrix}\right\}$$

on rappelle que $\dim(\operatorname{Im} f) = \operatorname{rg}(f)$
$= 3 - \dim(\operatorname{Ker} f)$
$= 3 - 1 = 2$

$$= \operatorname{Vect}\left\{\underbrace{\begin{pmatrix}-1\\1\\1\end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix}1\\2\\5\end{pmatrix}}_{u_2}\right\} = \operatorname{Vect}\left\{\underbrace{\begin{pmatrix}0\\3\\6\end{pmatrix}}_{u_1+u_2}, \underbrace{\begin{pmatrix}3\\0\\3\end{pmatrix}}_{u_2-2u_1}\right\}$$

$$= \operatorname{Vect}\left\{\begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\} = E_1 \quad \square$$

Autre méthode : $\operatorname{Im} f = \{(a,b,c) \in \mathbb{R}^3 \mid \exists (x,y,z) \in \mathbb{R}^3,\ \underbrace{f(x,y,z) = (a,b,c)}_{(S)}\}$

$$(S) \iff \begin{cases} 5x - 2y + z = 6a \\ -2x + 2y + 2z = 6b \\ x + 2y + 5z = 6c \end{cases} \iff \begin{cases} x + 2y + 5z = 6c & (L_3) \\ 6y + 12z = 6b + 12c & (L_2 + 2L_3) \\ -12y - 24z = 6a - 30c & (L_1 - 5L_3) \end{cases}$$

$$\iff \begin{cases} x + 2y + 5z = 6c \\ y + 2z = b + 2c & (L_2/6) \\ 2y + 4z = -a + 5c & (L_3/-6) \end{cases} \iff \begin{cases} x + 2y + 5z = 6c \\ y + 2z = b + 2c \\ 0 = -a + 5c - 2(b + 2c) \end{cases}$$

Soit

$$\operatorname{Im} f = \{(a,b,c) \in \mathbb{R}^3 \mid -a - 2b + c = 0\}$$
$$= \{(a, b, a+2b),\ a, b \in \mathbb{R}\} = \operatorname{Vect}\{(1,0,1), (0,1,2)\} = E_1$$

⑤
```
def ps(u,v):
    return sum([u[k]*v[k] for k in range(3)])
```

⑥ a) Si on a obtenu que :

$\mathrm{Im} f = \{(a,b,c) \in \mathbb{R}^3 \mid -a-2b+c=0\}$

alors il est immédiat que $\boxed{\mathrm{Im} f^{\perp} = \mathrm{Vect}\left\{\begin{pmatrix}-1\\-2\\1\end{pmatrix}\right\} = E_0}$

Sinon, on vérifie que $\langle(-1,-2,1),(1,0,1)\rangle = 0$
$= \langle(-1,-2,1),(0,1,2)\rangle$. ...

b) • $\forall x \in \mathbb{R}^3$, $f(x) \in \mathrm{Im} f = E_1$ donc $f(x) \in E_1$

• $f(f(x)-x) = f^2(x) - f(x) = 0$ car $A^2 = A$

D'où $f(x) - x \in \mathrm{Ker} f = \mathrm{Im} f^{\perp}$ et donc $\boxed{(f(x)-x \mid y) = 0 \quad \forall y \in E_1}$

**Conclusion** : $f$ est la projection orthogonale sur $E_1$

⑦ On rappelle que $d(t, E_1) = \|f(t) - t\|$

Il est facile de calculer $f(t) - t$ grâce à la matrice $A$ et on comprend mal pourquoi on demande une base orthonormée de $E_1$ ...

→ Je commence par la méthode rapide puis, page suivante, je respecte l'énoncé :

$$\mathcal{M}_B(f(t)-t) = A\begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}2\\4\\10\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1/3\\2/3\\5/3\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix}$$

D'où $f(t) - t = \left(-\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right)$ ; Appelons $u$ ce vecteur.

Alors

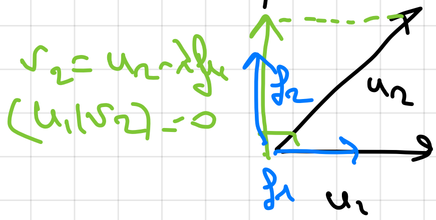
$$d(t, E_1) = \|u\| = \sqrt{\langle u, u\rangle}$$

$$= \texttt{np.sqrt(ps(u,u))}$$

**Conclusion** $\boxed{d(t, E_1) \simeq 1,633}$

⑦ Pour une méthode beaucoup plus longue...

$E_1 = \text{Vect}\{u_1, u_2\}$ où $u_1 = (1,0,1)$,
$u_2 = (0,1,2)$




$$f_1 = \frac{1}{\|u_1\|}\cdot u_1 = \frac{1}{\sqrt{2}}(1,0,1)$$

$$v_2 = u_2 - \lambda f_1 \quad (v_2|f_1) = 0$$

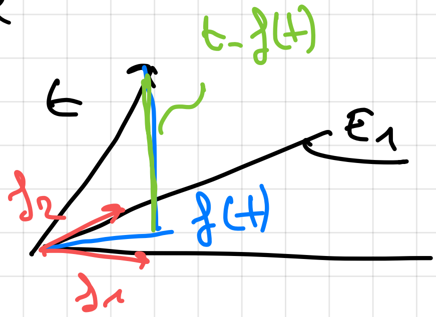
$$\Leftrightarrow (u_2|f_1) - \lambda(f_1|f_1) = 0 \Leftrightarrow \lambda = (u_2|f_1) = \frac{1}{\sqrt{2}}((0,1,2)|(1,0,1)) = \frac{2}{\sqrt{2}}$$

D'où $v_2 = u_2 - \frac{2}{\sqrt{2}} f_1$
$= (0,1,2) - (1,0,1) = (-1,1,1)$

et donc $f_2 = \frac{1}{\sqrt{3}}(-1,1,1)$.

$\boxed{B' = (f_1, f_2) \text{ est une b.o.n de } E_1}$

Utilisons maintenant le fait que
$t = (1,2,1)$ ; Alors ...




$$f(t) = (t|f_1)f_1 + (t|f_2)f_2$$
$$= \frac{1}{\sqrt{2}}\cdot 2 f_1 + \frac{2}{\sqrt{3}} f_2$$
$$= (1,0,1) + \frac{2}{3}(-1,1,1) = \left(\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right)$$

Dès lors,

$$d(t, E_1) = \|t - f(t)\| = \left\|\left(\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}\right)\right\|$$
$$= \sqrt{\frac{4}{9} + \frac{16}{9} + \frac{4}{9}} = \sqrt{\frac{24}{9}} = \boxed{\frac{2\sqrt{6}}{3}} \simeq 1,633.$$

□