$$B_4 = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$: ① λ vp de $B_4 \Leftrightarrow B_4 - \lambda I$ non inversible $\Leftrightarrow rg(B_4-\lambda I) < 4$

$$rg(B_4-\lambda I) = rg\begin{pmatrix} -\lambda & 1/2 & 0 & 0 \\ 1 & -\lambda & 1/2 & 0 \\ 0 & 1/2 & -\lambda & 0 \\ 0 & 0 & 1/2 & 1-\lambda \end{pmatrix} = rg\begin{pmatrix} \boxed{1} & -\lambda & 1/2 & 0 \\ 0 & 1/2 & -\lambda & 0 \\ 0 & 0 & 1/2 & 1-\lambda \\ -\lambda & 1/2 & 0 & 0 \end{pmatrix} \begin{matrix} L_1 \leftarrow L_2 \\ L_2 \leftarrow L_3 \\ L_3 \leftarrow L_4 \\ L_4 \leftarrow L_1 \end{matrix}$$

$$= rg\begin{pmatrix} 1 & -\lambda & 1/2 & 0 \\ 0 & \boxed{1/2} & -\lambda & 0 \\ 0 & 0 & 1/2 & 1-\lambda \\ 0 & 1/2-\lambda^2 & \lambda/2 & 0 \end{pmatrix} L_4 \leftarrow L_4 + \lambda L_1 = rg\begin{pmatrix} 1 & -\lambda & 1/2 & 0 \\ 0 & 1/2 & -\lambda & 0 \\ 0 & 0 & 1/2 & 1-\lambda \\ 0 & 0 & P(\lambda) & 0 \end{pmatrix} \begin{matrix} L_4 \leftarrow (\frac{1}{2}-\lambda^2)L_2 \\ - 1/2 L_4 \end{matrix}$$

avec $P(\lambda) = -\lambda(1/2 - \lambda^2) - \frac{\lambda}{4} = \lambda\left[\lambda^2 - \frac{1}{2} - \frac{1}{4}\right] = \lambda\left(\lambda^2 - \frac{3}{4}\right)$

(!)

$$= rg\begin{array}{c} \begin{matrix} x & y & t & z \end{matrix} \\ \begin{pmatrix} 1 & -\lambda & 0 & 1/2 \\ 0 & 1/2 & 0 & -\lambda \\ 0 & 0 & 1-\lambda & 1/2 \\ 0 & 0 & 0 & P(\lambda) \end{pmatrix} \end{array} \quad \underline{C_3 \leftrightarrow C_4.} = rg(U_\lambda).$$

<u>Concl.</u> : λ vp de $B_4 \Leftrightarrow P(\lambda) = 0 \Leftrightarrow \lambda \in \{0, -\sqrt{3}/2, \sqrt{3}/2, 1\}$.

② $X \in E_1 \Leftrightarrow (B_4 - I)X = 0 \Leftrightarrow U_1 X = 0 \Leftrightarrow \begin{cases} x - y + 1/2\, z = 0 \\ 1/2\, y - z = 0 \\ 1/2\, z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 = y \\ = z \\ \forall t \in \mathbb{R}. \end{cases}$

Donc $\underline{E_1 = Vect\{(0,0,0,1)\}}$.

$X \in E_0 \Leftrightarrow B_4 X = 0 \Leftrightarrow U_0 X = 0 \Leftrightarrow \begin{cases} x + z/2 = 0 \\ 1/2\, y = 0 \\ t + z/2 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = -z/2 \\ t = -z/2 \end{cases} \forall z \in \mathbb{R}$

Donc $\underline{E_0 = Vect\{(1,0,-2,1)\}}$

$X \in E_{\sqrt{3}/2} \Leftrightarrow (B_4 - \frac{\sqrt{3}}{2} I)X = 0 \Leftrightarrow U_{\sqrt{3}/2} X = 0 \Leftrightarrow \begin{cases} x - \sqrt{3}/2\, y + z/2 = 0 \\ 1/2\, y - \sqrt{3}/2 \cdot z = 0 \\ (1-\sqrt{3}/2)t + z/2 = 0 \end{cases}$

Donc $E_{\sqrt{3}/2} = Vect\{(1, \sqrt{3}, 1, \frac{1}{\sqrt{3}-2})\}$

$= \underline{Vect\{(1, \sqrt{3}, 1, -\sqrt{3}-2)\}}$

De même : $E_{-\sqrt{3}/2} = Vect\{(1, -\sqrt{3}, 1, -\frac{1}{2+\sqrt{3}})\} = \underline{Vect\{(1, -\sqrt{3}, 1, \sqrt{3}-2)\}}$.

③ $Card[Sp(B_4)] = 4 = ordre(B_4)$ <u>Donc</u> B_4 est diagonalisable.

avec $B_4^n = P D^n P^{-1}$ où $P = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & \sqrt{3} & -\sqrt{3} \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -\sqrt{3}-2 & \sqrt{3}-2 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1/3 & 0 & -1/3 & 0 \\ 1/3 & \sqrt{3}/6 & 1/6 & 0 \\ 1/3 & -\sqrt{3}/6 & 1/6 & 0 \end{pmatrix}$

Soit

$$B_4^n = \begin{pmatrix} 1/3\left[(\sqrt{3}/2)^n + (-\sqrt{3}/2)^n\right] & \frac{\sqrt{3}}{6}\left((\frac{\sqrt{3}}{2})^n - (-\sqrt{3}/2)^n\right) & \cdots & 0 \\ \sqrt{3}/2\left[(\sqrt{3}/2)^n - (-\sqrt{3}/2)^n\right] & \cdots & \cdots & 0 \\ 1/3\left[(\sqrt{3}/2)^n + (-\sqrt{3}/2)^n\right] & \cdots & \cdots & 0 \\ 1 + (-\sqrt{3}-2)(\sqrt{3}/2)^n + (\sqrt{3}-2)(-\frac{\sqrt{3}}{2})^n & \cdots & \cdots & 1 \end{pmatrix}$$