$$B_2 = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

① λ vp de $B_2 \iff B_2 - \lambda I_3$ non inversible $\iff \operatorname{rg}(B_2 - \lambda I_3) < 3$

or $\operatorname{rg}(B_2 - \lambda I) = \operatorname{rg}\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 4 & 2 & 3-\lambda \\ 2 & -\lambda & 2 \\ 3-\lambda & 2 & 4 \end{pmatrix} L_1 \leftrightarrow L_3$

$$= \operatorname{rg}\begin{pmatrix} 4 & 2 & 3-\lambda \\ 0 & -2(\lambda+1) & \lambda+1 \\ 0 & 2(\lambda+1) & P(\lambda) \end{pmatrix} \begin{matrix} L_2 \leftarrow 2L_2 - L_1 \\ L_3 \leftarrow 4L_3 - (3-\lambda)L_1. \end{matrix}$$

avec $P(\lambda) = 16 - (3-\lambda)^2 = (4-3+\lambda)(4+3-\lambda) = (\lambda+1)(7-\lambda)$

$$= \operatorname{rg}\begin{pmatrix} 4 & 2 & 3-\lambda \\ 0 & -2(\lambda+1) & \lambda+1 \\ 0 & 0 & Q(\lambda) \end{pmatrix} L_3 \leftarrow L_3 + L_2 \quad = \operatorname{rg}(U_\lambda)$$

où $Q(\lambda) = P(\lambda) + \lambda + 1 = (\lambda+1)(7-\lambda) + (\lambda+1)$ et U_λ la réduite de Gauss de $B_2 - \lambda I$

$= (\lambda+1)(8-\lambda)$

Conclusion $\boxed{\operatorname{Sp}(B_2) = \{-1, 8\}.}$

② $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_{-1} \iff (B_2 + I_3)X = 0 \iff U_{-1}.X = 0 \iff 4x + 2y + 4z = 0$ $\iff 2x + y + 2z = 0.$

$E_{-1} = \left\{ \begin{pmatrix} x \\ -2x-2z \\ z \end{pmatrix} ; x, z \in \mathbb{R} \right\} = \operatorname{Vect}\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$

$X \in E_8 \iff (B_2 - 8I_3)X = 0 \iff U_8.X = 0 \iff \begin{cases} 4x + 2y - 5z = 0 \\ -2y + z = 0 \end{cases}$

$\iff \begin{cases} z = 2y & (L_2) \\ x = 2y & (L_1) \end{cases}$

$E_8 = \left\{ \begin{pmatrix} 2y \\ y \\ 2y \end{pmatrix}, y \in \mathbb{R} \right\} = \operatorname{Vect}\left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$; $\dim E_{-1} + \dim E_8 = 2 + 1$ $= 3 = \text{ordre}(B_2)$

$\Rightarrow$ B_2 diagonalisable.

$B_2 = PDP^{-1}$ où $P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & -2 \\ 2 & 0 & 1 \end{pmatrix}$; $D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

On obtient P^{-1} en résolvant $PX' = X \iff X' = P^{-1}X$: $P^{-1} = \frac{1}{9}\begin{pmatrix} 2 & 1 & 2 \\ 5 & -2 & -4 \\ -4 & -2 & 5 \end{pmatrix}$

D'où (Récurrence),

$$B_2^n = PD^nP^{-1} = \frac{1}{9}\begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & -2 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 2 & 1 & 2 \\ 5 & -2 & -4 \\ -4 & -2 & 5 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 2\cdot 8^n & (-1)^n & 0 \\ 8^n & -2(-1)^n & -2(-1)^n \\ 2\cdot 8^n & 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 2 & 1 & 2 \\ 5 & -2 & -4 \\ -4 & -2 & 5 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 4\cdot 8^n + 5(-1)^n & 2\cdot 8^n - 2(-1)^n & 4\cdot 8^n - 4(-1)^n \\ 2\cdot 8^n - 2(-1)^n & 8^n + 8(-1)^n & 2\cdot 8^n - 2(-1)^n \\ 4\cdot 8^n - 4(-1)^n & 2\cdot 8^n - 2(-1)^n & 4\cdot 8^n + 5(-1)^n \end{pmatrix}$$